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Floquet Energies and Quantum Hall Effect in a Periodic Potential

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The Quantum Hall Effect for free electrons in external periodic potential is discussed without using the linear response approximation. We find that the Hall conductivity is related in a simple way to Floquet energies (associated to the Schrödinger equation in the co-moving frame). By this relation one can analyze the dependence of the Hall conductivity from the electric field. Sub-bands can be introduced by the time average of the expectation value of the Hamiltonian on the Floquet states. Moreover we prove previous results in form of sum rules as, for instance: the topological character of the Hall conductivity (being an integer multiple of e^2/h), the Diofantine equation which constrains the Hall conductivity by the rational number which measures the flux of the magnetic field through the periodicity cell. The Schrödinger equation fixes in a natural way the phase of the wave function over the reduced Brillouin zone: thus the topological invariant providing the Hall conductivity can be evaluated numerically without ambiguity.

I. INTRODUCTION AND CONCLUSIONS

A milestone in the theory of the Quantum Hall Effect is the result obtained by Thouless *et al.* [1] concerning the Hall conductivity $\sigma_{\rm H}$ for free electrons in a periodic potential. Their proof showed that $\sigma_{\rm H}$ is a multiple integer of e^2/h, if the chemical potential lies in a gap of the Hofstadter spectrum [2] (gap condition). The result is rather striking, since the coefficient varies strongly with the number of filled bands and with the commensurability factor q/p, which gives the flux of the magnetic field through the periodicity cell $\Phi = \frac{p}{q}\frac{hc}{e}$. Subsequent works [3–10] have shown the generality of the result by evidencing the topological nature of the Hall conductivity. Moreover the result has been shown to be valid also in presence of many-body interaction, provided the ground state is not degenerate [6–8]

The proof by Thouless *et al.* makes use of the linear response approximation (Kubo formula). To our knowledge all works devoted to this problem use the linear approximation or, equivalently, the adiabatic approximation.

There are few reasons that make the case of *finite* electric field an interesting problem. First, from the experimental point of view it may be interesting to observe the phenomenon when the electric field varies [11]. Second, the limit of weak periodic potential is in conflict with the limit of small electric field. It is important to investigate the intermediate situation where the potential is weak in comparison to the Landau splitting and comparable to the electrostatic potential.

The present work deals with the problem without using the linear response approximation. We consider a Galilei transformation in order to have the time dependence of the Hamiltonian only in the external periodic potential. The Hamiltonian, being periodic both under magnetic and time translations, allows an analysis in terms of Bloch functions and Floquet eigenstates [16]. Thus the

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dynamical problem consists in finding the quasi-periodic solutions of the Schrödinger equation (Floquet states) and their quasi-energies (Floquet energies).

It is shown that there is a simple relation between the Hall conductivity and the Floquet energies associated to the periodic Hamiltonian. Floquet energies can be easily obtained by diagonalization the time evolution operator. The average in time over a period of the expectation value of the Hamiltonian over Floquet states turns out to be a convenient quantity in order to define *sub-bands* on the reduced Brillouin zone. A strong electric field complicates the construction of a stationary state, since it induces transitions to energies above the chemical potential. In fact a spectrum of energy, which satisfies the gap condition in the adiabatic approximation, might appear partially unresolved for finite values of the electric field.

If the gap condition is satisfied, then the Hall conductivity is an integer and moreover it is ruled by a Diofantine equation [1,17–19]. These results are valid as sum rules, i.e. in the case where a group of sub-bands cross with each other.

In the adiabatic approximation there is a simple relation between Floquet energies and Berry phases. Moreover the phase of the instantaneous eigenvector can be chosen in an essentially unique way [5]. Thus the Hall conductivity can be evaluated by means of a suitable line integral on the border of the reduced Brillouin zone $\mathcal{C}$ [1,12]

$$-\frac{i}{2}\int_{\partial\mathcal{C}} dw_l \big(\langle\frac{\partial}{\partial w_l}\Upsilon_J^{(w)}|\Upsilon_J^{(w)}\rangle - \langle\Upsilon_J^{(w)}|\frac{\partial}{\partial w_l}\Upsilon_J^{(w)}\rangle\big) \tag{1.1}$$

(see eq. 7.5).

II. THE MODEL

The Schrödinger equation in presence of a magnetic field B (along z, in the symmetric gauge) and of an electric field $\mathbf{E}$ is [13]

$$\begin{aligned} &i\partial_t\psi = H\psi \\ &H = \frac{1}{2}\big[(-i\partial_1 - \frac{r_2}{2} - \mathcal{E}_1 t)^2 + (-i\partial_2 + \frac{r_1}{2} - \mathcal{E}_2 t)^2\big] + \mathcal{V}(\mathbf{r}) \end{aligned} \tag{2.1}$$

where

$$\mathcal{E}_i = \frac{e\lambda E_i}{\hbar\omega}, \qquad \mathcal{V}(\mathbf{r}) = \frac{1}{\hbar\omega}V(\lambda\mathbf{r}). \tag{2.2}$$

The external potential is periodic over a lattice

$$\mathcal{V}(\mathbf{r} + m\mathbf{c}' + n\mathbf{d}') = \mathcal{V}(\mathbf{r}) \quad \forall(m,n)\in\mathcal{Z}^2. \tag{2.3}$$

The flux through the cell is given by a rational number of the quantum unit of flux

$$\tilde{\mathbf{c}}'\cdot\mathbf{d}' = 2\pi\frac{p}{q}. \tag{2.4}$$

Thus we introduce a convenient sub-lattice $(\mathbf{c},\mathbf{d})$ and a finite domain with area A and side-vectors $(\mathbf{L}_1,\mathbf{L}_2)$ where the electrons live. The commensurability is given by

$$\begin{cases} \mathbf{c} = r\mathbf{c}' + s'\mathbf{d}' \\ \mathbf{d} = r'\mathbf{c}' + s\mathbf{d}' \\ r,s,r',s' \ \text{ integers} \\ rs - r's' = q \end{cases} \qquad \begin{cases} \mathbf{L}_1 = k\mathbf{c} + l'\mathbf{d} \\ \mathbf{L}_2 = k'\mathbf{c} + l\mathbf{d} \\ k,l,k',l' \ \text{ integers} \\ kl - k'l' = N \equiv \frac{g_L}{p} \end{cases} \tag{2.5}$$

where $g_L = A/(2\pi)$ is the degeneracy of the (unperturbed) Landau level. The boundary conditions for eq. (2.1) are imposed by means of the Magnetic Translation operator [14,10]

$$S(v) \equiv \exp(\frac{i}{2}\tilde{\mathbf{v}} \cdot \mathbf{r}) \exp(v_i \partial_i) \tag{2.6}$$

over the domain of definition given by the parallelogram with side vectors $\mathbf{L}_1, \mathbf{L}_2$. We consider periodic boundary conditions

$$\begin{aligned} S(L_1)\psi &= e^{i\theta_1}\psi \\ S(L_2)\psi &= e^{i\theta_2}\psi. \end{aligned} \tag{2.7}$$

Both operators $S(c)$ and $S(d)$ commute with the Hamiltonian and moreover [15]

$$[S(c), S(d)] = 0. \tag{2.8}$$

Then the solutions of eq. (2.1) can be labeled by the phases μ, ν given by

$$\begin{aligned} S(c)\psi^{\mu\nu} &= e^{i\mu}\psi^{\mu\nu} \\ S(d)\psi^{\mu\nu} &= e^{i\nu}\psi^{\mu\nu}. \end{aligned} \tag{2.9}$$

The values of μ, ν are fixed by the conditions (2.5) and (2.7) in a standard way.

III. GALILEI TRANSFORMATION AND BLOCH FUNCTIONS

We consider a Galilei transformation which removes the electric field from the kinetic term in eq. (2.1). We use the operator [14]

$$T(v) \equiv \exp(-\frac{i}{2}\tilde{\mathbf{v}} \cdot \mathbf{r}) \exp(v_i \partial_i) \tag{3.1}$$

which commutes with any operator S. The unitary transformation

$$\psi^{(G)} = T(-(\mathcal{E} + \tilde{\mathcal{E}}t))\psi \tag{3.2}$$

yields a function which satisfies a Schrödinger equation with Hamiltonian

$$\begin{aligned} H^{(G)} = &\frac{1}{2}\big[(-i\partial_1 - \frac{r_2}{2})^2 + (-i\partial_2 + \frac{r_1}{2})^2\big] \\ &+\mathcal{V}(r - (\mathcal{E} + \tilde{\mathcal{E}}t)). \end{aligned} \tag{3.3}$$

$\psi^{(G)}$ can be chosen to satisfy the conditions (2.9).

The phases μ, ν assume a finite number of values, proportional to the area A of the domain. Eventually we will extrapolate to continuous values. However this cannot be done directly on $\psi^{\mu\nu}$ (eq. (2.9)), without running into the following paradox. Take the derivative respect to μ and then the expectation value on $\psi^{\mu\nu}$

$$\langle \psi^{\mu\nu}, S(c)\frac{\partial}{\partial\mu}\psi^{\mu\nu}\rangle = ie^{i\mu}\|\psi^{\mu\nu}\|^2 + e^{i\mu}\langle \psi^{\mu\nu}, \frac{\partial}{\partial\mu}\psi^{\mu\nu}\rangle. \tag{3.4}$$

Then use the unitarity for $S(c)$

$$\langle S(-c)\psi^{\mu\nu}, \frac{\partial}{\partial\mu}\psi^{\mu\nu}\rangle = e^{i\mu}\langle \psi^{\mu\nu}, \frac{\partial}{\partial\mu}\psi^{\mu\nu}\rangle. \tag{3.5}$$

From eqs. (3.4) and (3.5) one gets

$$\|\psi^{\mu\nu}\| = 0. \tag{3.6}$$

This result is absurd.

This difficulty can be avoided by introducing the unitary equivalent functions

$$\Upsilon^{(w_{\mu\nu})} \equiv S(-w_{\mu\nu})\psi^{(G)\mu\nu} \tag{3.7}$$

where

$$w_{\mu\nu} = \frac{1}{2\pi p}[(\nu-\nu_0)c - (\mu-\mu_0)d] \tag{3.8}$$

(μ_0, ν_0 are fixed). The corresponding Hamiltonian is

$$H^{(\Upsilon)}_{w_{\mu\nu}} = \frac{1}{2}\big[(-i\partial_1 - \frac{r_2}{2})^2 + (-i\partial_1 + \frac{r_1}{2})^2\big] + \mathcal{V}(r - w_{\mu\nu} - (\mathcal{E} + \tilde{\mathcal{E}}t)). \tag{3.9}$$

$\Upsilon^{(w_{\mu\nu})}$ are like Bloch functions. In fact, by using the composition law of S

$$S(v)S(w) = S(v+w)\exp(-\frac{i}{2}\tilde{\mathbf{v}}\cdot\mathbf{w}), \tag{3.10}$$

one can easily check that

$$\begin{aligned} S(c)\Upsilon^{(w_{\mu\nu})} &= e^{i\mu_0}\Upsilon^{(w_{\mu\nu})} \\ S(d)\Upsilon^{(w_{\mu\nu})} &= e^{i\nu_0}\Upsilon^{(w_{\mu\nu})}, \end{aligned} \tag{3.11}$$

i.e. the boundary conditions on the periodicity lattice are fixed. This fact makes the extrapolation to continuous values of $w_{\mu\nu}$ harmless. In particular the unitarity property of operators S is preserved.

The reduced Brillouin zone is given by the domain containing all possible values of $w_{\mu\nu}$

$$\mathcal{C} \equiv \left\{\mathbf{w} \in \mathcal{R}^2 : \mathbf{w} = \frac{1}{p}(\lambda_1\mathbf{c} + \lambda_2\mathbf{d}), \quad 0 < \lambda_j < 1 \quad j = 1,2\right\}. \tag{3.12}$$

If $\tau\tilde{\mathcal{E}}$ is a site of the lattice $\mathbf{c}', \mathbf{d}'$, the Hamiltonian $H^{(\Upsilon)}_{w_{\mu\nu}}$ is periodic in time with period τ. Thus we assume

$$\tau\tilde{\mathcal{E}} = m_0\mathbf{c}' + n_0\mathbf{d}' \qquad (m_0, n_0 \text{ integers}). \tag{3.13}$$

For technical reasons we introduce another set of (unitary equivalent) functions and the corresponding Hamiltonian ($T(w)$ exists for any vector w [15])

$$\Xi^{(w)} \equiv T(w)\Upsilon^{(w)} \tag{3.14}$$

$$H^{(\Xi)}_{w} = \frac{1}{2}\big[(-i\partial_1 - \frac{r_2}{2} - w_2)^2 + (-i\partial_1 + \frac{r_1}{2} + w_1)^2\big] + \mathcal{V}(r - (\mathcal{E} + \tilde{\mathcal{E}}t)). \tag{3.15}$$

IV. THE HALL CURRENT

The stationary state for a system with time-dependent Hamiltonian at zero temperature (in the present case it is a good approximation) can be defined by a minimum criterion. For any change on the stationary state

$$\Delta\left\{\frac{1}{2t_0}\int_{-t_0}^{t_0} dt\langle H^{(\Upsilon)}(t)\rangle - \epsilon_F\langle N_e\rangle\right\} \geq 0 \tag{4.1}$$

with $t_0 >> \tau$ and where ϵ_F is the chemical potential and N_e is the operator that counts the number of electrons.

The space average of the Hall current is (in units $e\hbar/(m\lambda^3)$)

$$J_1 = \frac{1}{(2\pi)^2}\int_{\mathcal{C}} d^2w \sum_J{}' \langle \Upsilon_J^{(w)}|[p_1 - \frac{r_2}{2} + \mathcal{E}_2]|\Upsilon_J^{(w)}\rangle$$
$$J_2 = \frac{1}{(2\pi)^2}\int_{\mathcal{C}} d^2w \sum_J{}' \langle \Upsilon_J^{(w)}|[p_2 + \frac{r_1}{2} - \mathcal{E}_1]|\Upsilon_J^{(w)}\rangle. \tag{4.2}$$

The index J labels a complete set of normalized solutions of the Schrödinger equation. In Sections V and VI we provide a choice for the orthonormal set by constructing sub-bands. The conditioned sum is over those states which satisfy the condition in eq. (4.1). In general the sum over J is discontinuous in w. In terms of Ξ functions the current is

$$J_i = -\epsilon_{ij}\frac{1}{(2\pi)^2}\int_{\mathcal{C}} d^2w \sum_J{}' \langle \Xi_J^{(w)}|\frac{\partial H_w^{(\Xi)}}{\partial w_j}|\Xi_J^{(w)}\rangle + \frac{\nu_f}{2\pi}\epsilon_{ij}\mathcal{E}_j \tag{4.3}$$

where ν_f is the filling factor. By using the Schrödinger equation it can be written as

$$J_i = -i\epsilon_{ij}\frac{1}{(2\pi)^2}\int_{\mathcal{C}} d^2w \sum_J{}' \partial_t[\langle \Xi_J^{(w)}|\partial_j|\Xi_J^{(w)}\rangle] + \frac{\nu_f}{2\pi}\epsilon_{ij}\mathcal{E}_j. \tag{4.4}$$

The dependence from wave function $\Upsilon_J^{(w)}$ can be obtained by introducing the creation and annihilation operators $a, a^\dagger$

$$\partial_1 T(w) = T(w)\Big[\frac{1}{\sqrt{2}}(a - a^\dagger) + \frac{i}{2}w_2\Big]$$
$$\partial_2 T(w) = T(w)\Big[\frac{i}{\sqrt{2}}(a + a^\dagger) - \frac{i}{2}w_1\Big]. \tag{4.5}$$

We get

$$J_1 = \frac{\nu_f}{2\pi}\mathcal{E}_2 + \frac{1}{(2\pi)^2}\int_{\mathcal{C}} d^2w \sum_J{}' \partial_t\Big\{\langle \Upsilon_J^{(w)}|[\frac{1}{\sqrt{2}}(a + a^\dagger) - \frac{1}{2}w_1]|\Upsilon_J^{(w)}\rangle$$
$$- i\langle \Upsilon_J^{(w)}|\partial_2\Upsilon_J^{(w)}\rangle\Big\}$$
$$J_2 = -\frac{\nu_f}{2\pi}\mathcal{E}_1 + \frac{1}{(2\pi)^2}\int_{\mathcal{C}} d^2w \sum_J{}' \partial_t\Big\{\langle \Upsilon_J^{(w)}|[\frac{i}{\sqrt{2}}(a - a^\dagger) - \frac{1}{2}w_2]|\Upsilon_J^{(w)}\rangle$$
$$+ i\langle \Upsilon_J^{(w)}|\partial_1\Upsilon_J^{(w)}\rangle\Big\}. \tag{4.6}$$

We consider now the time average of the density of current over a period. If the wave functions obey the Floquet condition

$$\Upsilon_J^{(w)}(t + \tau) = \exp\{-iE_J^{(w)}\}\Upsilon_J^{(w)}(t) \tag{4.7}$$

then

$$\langle J_i\rangle = \epsilon_{ij}\frac{1}{2\pi}\Big[\nu_f\mathcal{E}_j - i\frac{1}{2\pi}\int_{\mathcal{C}} d^2w \sum_J{}' \frac{1}{\tau}\int_0^\tau dt\partial_t\langle \Upsilon_J^{(w)}|\partial_j\Upsilon_J^{(w)}\rangle\Big]. \tag{4.8}$$

Now we take the components parallel and orthogonal to the electric field and get the Hall conductivity (in units e^2/h)

$$\sigma_{xx} \equiv \frac{\mathcal{E}_i}{\mathcal{E}^2}\langle J_i\rangle$$
$$= -i\epsilon_{ik}\frac{\mathcal{E}_i}{\mathcal{E}^2}\frac{1}{2\pi}\int_{\mathcal{C}} d^2w \sum_J{}' \frac{1}{\tau}\int_0^\tau dt\partial_t\langle \Upsilon_J^{(w)}|\partial_k\Upsilon_J^{(w)}\rangle \tag{4.9}$$
$$\sigma_{xy} \equiv \epsilon_{ik}\frac{\mathcal{E}_k}{\mathcal{E}^2}\langle J_i\rangle$$
$$= \Big[\nu_f - \frac{i}{2\pi}\int_{\mathcal{C}} d^2w \sum_J{}' \frac{1}{\tau}\int_0^\tau dt\partial_t\frac{\mathcal{E}_k}{\mathcal{E}^2}\langle \Upsilon_J^{(w)}|\partial_k\Upsilon_J^{(w)}\rangle\Big]. \tag{4.10}$$

Since eq. (4.7) is valid for any t, the Hall conductivity in eqs. (4.9) and (4.10) is constant in time.

V. FLOQUET ENERGY

The time evolution of the wave functions $\Upsilon^{(w)}$ is given by the Schrödinger equation associated to the Hamiltonian in eq. (3.9) with the boundary conditions (3.11). Consider the unitary evolution operator $U^{(w)}(t,t')$ which satisfies

$$i\partial_t U^{(w)}(t,t') = H_w^{(\Upsilon)} U^{(w)}(t,t') \qquad U^{(w)}(t',t') = 1. \tag{5.1}$$

The solution of eq. (5.1) is

$$U(t,s) = \sum_{n=0,\infty} \frac{(-i)^n}{n!} \int_s^t dt_1 \dots dt_n T \left(H_w^{(\Upsilon)}(t_1) \dots H_w^{(\Upsilon)}(t_n) \right). \tag{5.2}$$

The solutions of the Schrödinger equation, that are quasi-periodic in time (eq. (4.7)), are both right and (complex conjugate) left eigenvectors of $U^{(w)}(t'+\tau,t')$. The eigenvalues

$$\exp(-iE_J^{(w)}) \tag{5.3}$$

provide the Floquet energies $E_J^{(w)}$ [mod(2π)] [16].

In the adiabatic limit the Floquet energies are given by

$$E_J^{(w)} = \int_0^\tau dt \lambda_J(w + \tilde{\mathcal{E}} t) + \alpha_J^{(w)} \tag{5.4}$$

where $\lambda_J(w)$ is the instantaneous eigenvalue of the Hamiltonian and $\alpha_J^{(w)}$ is the Berry phase [1,12].

One can easily verify that

$$S(-\frac{c}{p}) U^{(w)}(t'+\tau,t') S(\frac{c}{p}) = U^{(w+\frac{c}{p})}(t'+\tau,t'), \tag{5.5}$$

then the set of eigenvalues is the same

$$\left\{ \exp(-iE_J^{(w)}) \right\} = \left\{ \exp(-iE_J^{(w+\frac{c}{p})}) \right\} \quad \forall w. \tag{5.6}$$

Similarly one gets

$$\left\{ \exp(-iE_J^{(w)}) \right\} = \left\{ \exp(-iE_J^{(w+\frac{d}{p})}) \right\} \quad \forall w. \tag{5.7}$$

Thus we get

$$\begin{aligned} E_J^{(w)} &= E_{J_c}^{(w+\frac{c}{p})} + 2\pi k_{Jc} \\ E_J^{(w)} &= E_{J_d}^{(w+\frac{d}{p})} + 2\pi k_{Jd} \quad \forall w \end{aligned} \tag{5.8}$$

where k_{Jc}, k_{Jd} are integers. The one-to-one mapping $J \to J_c$ and $J \to J_d$ depends strongly on the direction and on the strength of the electric field. A further consequence of eq. (5.5) is that

$$\begin{aligned} \Upsilon_J^{(w)} &= \exp(i\phi_J(w)) S(\frac{c}{p}) \Upsilon_{J_c}^{(w+\frac{c}{p})} \\ \Upsilon_J^{(w)} &= \exp(i\psi_J(w)) S(\frac{d}{p}) \Upsilon_{J_d}^{(w+\frac{d}{p})} \quad \forall w \end{aligned} \tag{5.9}$$

where $\phi_J(w), \psi_J(w)$ are phases.

The periodic time dependence of the Hamiltonian gives further informations about the Floquet energies

$$U^{(w)}(t-t',s-t') = U^{(w+\tilde{\mathcal{E}}t')}(t,s). \tag{5.10}$$

Therefore the Floquet energies are constant along the line $w+\tilde{\mathcal{E}}t$

$$\frac{\partial}{\partial t} E_J^{(w+\tilde{\mathcal{E}}t)} = 0. \tag{5.11}$$

This implies that the longitudinal Hall conductivity σ_{xx} in eq. (4.9) is zero. Moreover for the transverse Hall conductivity the reduced Brillouin zone can be chosen to be a rectangular $\mathcal{C}'$ with side vectors laying on $\mathcal{E}$ and $\tilde{\mathcal{E}}$. Either $\mathbf{c}/p$ (or $\mathbf{d}/p$) lies on the side $\mathcal{E}$. Then a simple inspection to the geometry gives

$$\sigma_{xy} = \Big[\nu_f - \frac{1}{\tau \mathbf{c}\cdot\mathcal{E}} \int_\Gamma dw_k \sum_J{}' \partial_k E_J^{(w)}\Big] \tag{5.12}$$

with

$$\tau \mathbf{c}\cdot\mathcal{E} = 2\pi \frac{p}{q}(n_0 r - m_0 s'). \tag{5.13}$$

The line integral Γ is the straight segment along the electric field, starting in $p^{-1}[\mathbf{c} - \mathcal{E}^{-2}(\mathbf{c}\cdot\mathcal{E})\mathcal{E}]$ and ending in $\mathbf{c}/p$. Equation (5.12) provides a direct way to evaluate the Hall conductivity. One has to find the Floquet energies and then the sub-bands have to be reconstructed by requiring continuity on the reduced Brillouin zone. If the gap condition is satisfied then any sub-band is either filled or empty. As a consequence of this situation we get

$$\int_\Gamma dw_k \sum_J{}' \partial_k E_J^{(w)} = \sum_J{}' \int_\Gamma dw_k \partial_k E_J^{(w)} = \sum_J{}' \Big(E_J^{(\frac{c}{p})} - E_J^{(0)}\Big). \tag{5.14}$$

The periodicity of the Hamiltonian in w across $\mathbf{c}', \mathbf{d}'$ in eq. (3.9) implies relations similar to those in eqs. (5.6) and (5.7)

$$\begin{aligned} \Big\{\exp(-iE_J^{(w)})\Big\} &= \Big\{\exp(-iE_J^{(w+c')})\Big\} \\ \Big\{\exp(-iE_J^{(w)})\Big\} &= \Big\{\exp(-iE_J^{(w+d')})\Big\} \quad \forall w. \end{aligned} \tag{5.15}$$

They imply

$$\begin{aligned} E_J^{(w)} &= E_{J'_c}^{(w+c')} + 2\pi k'_{Jc} \\ E_J^{(w)} &= E_{J'_d}^{(w+d')} + 2\pi k'_{Jd} \quad \forall w. \end{aligned} \tag{5.16}$$

Again the mapping among Floquet energies is generally non-trivial.

VI. SUB-BANDS, CLUSTERS AND SUM RULES

The sub-bands are surfaces given by

$$\mathcal{I}_J^{(w)} \equiv \frac{1}{\tau}\int_0^\tau dt \langle \Upsilon_J^{(w)}(t)|H_w^{(\Upsilon)}(t)|\Upsilon_J^{(w)}(t)\rangle \tag{6.1}$$

where w is any point in the reduced Brillouin zone and J denotes the sub-band. The raw data given by the eigenvalues and eigenvectors of $U(\tau,0)$ have to be organized so that the sub-bands are regular surfaces. Figs. 1-10 give examples of this procedure. Although for a finite strength of the electric

field the sub-bands described by $\mathcal{I}_J(w)$ are complicated surfaces which cross each other, in general the construction of a sub-band shows no difficulties since the Floquet energies $E_J(w)$ appear to be smooth functions. Some properties are helpful for their construction. Eq. (5.11) tells that sub-bands vary only in the direction given by the electric field. Moreover eqs. (5.8) and (5.9) give bounds on the boundary of the reduced Brillouin zone. As expected, in the adiabatic limit one observes the eigenvalue patterns of the instantaneous Hamiltonian (their time averages). As the strength of the electric field rises, the mixing of these eigenvalues to form a sub-band becomes more and more complicated. Sometime the construction of the sub-band is difficult due to a pinching process on two (or more) lines. This process is exemplified in Figs. 5 and 9. In particular Fig. 9 shows that as the electric field decreases the v-like lines merge to form x-crossing regular lines. At the same time a gap appears as in Fig. 3.

We consider here the situation where the sub-bands flock in clusters separated by gaps. Then the mapping of the Floquet energies in w to those in $w+\frac{c}{p}$, $w+\frac{d}{p}$, $w+c'$ and $w+d'$ is limited to the set separated by gaps. A sub-band has the time average of the energy that crosses at least one of the other elements of the set, but none of those outside. Thus eqs. (5.8) and (5.16) provide the sum rules for every cluster $\mathcal{S}$

$$\begin{aligned}
\sum_{J\in\mathcal{S}} E_J^{(w+\frac{c}{p})} &= \sum_{J_c\in\mathcal{S}} E_{J_c}^{(w)} + 2\pi\mathcal{K}_{\mathcal{S}c} \qquad \mathcal{K}_{\mathcal{S}c} \equiv \sum_{J\in\mathcal{S}} k_{Jc} \\
\sum_{J\in\mathcal{S}} E_J^{(w+\frac{d}{p})} &= \sum_{J_d\in\mathcal{S}} E_{J_d}^{(w)} + 2\pi\mathcal{K}_{\mathcal{S}d} \qquad \mathcal{K}_{\mathcal{S}d} \equiv \sum_{J\in\mathcal{S}} k_{Jd}
\end{aligned} \tag{6.2}$$

and

$$\begin{aligned}
\sum_{J\in\mathcal{S}} E_J^{(w+c')} &= \sum_{J'_c\in\mathcal{S}} E_{J'_c}^{(w)} + 2\pi\mathcal{K}'_{\mathcal{S}c} \qquad \mathcal{K}'_{\mathcal{S}c} \equiv \sum_{J\in\mathcal{S}} k'_{Jc} \\
\sum_{J\in\mathcal{S}} E_J^{(w+d')} &= \sum_{J'_d\in\mathcal{S}} E_{J'_d}^{(w)} + 2\pi\mathcal{K}'_{\mathcal{S}d} \qquad \mathcal{K}'_{\mathcal{S}d} \equiv \sum_{J\in\mathcal{S}} k'_{Jd}.
\end{aligned} \tag{6.3}$$

The contribution of a filled set of sub-bands to the Hall conductivity is then by eq. (5.14)

$$\sigma_{xy} = \nu_f - \frac{2\pi}{\tau\mathbf{c}\cdot\mathcal{E}} \sum_{\mathcal{S}}{}' \mathcal{K}_{\mathcal{S}c}. \tag{6.4}$$

The periodicity in time expressed by eq. (3.13) implies (see also eq. (5.11))

$$m_0\mathcal{K}'_{\mathcal{S}c} + n_0\mathcal{K}'_{\mathcal{S}d} = 0. \tag{6.5}$$

Since m_0, n_0 are relative prime numbers then an integer $\mathcal{K}'_{\mathcal{S}}$ exists such that

$$\begin{aligned}
\mathcal{K}'_{\mathcal{S}c} &= n_0\mathcal{K}'_{\mathcal{S}} \\
\mathcal{K}'_{\mathcal{S}d} &= -m_0\mathcal{K}'_{\mathcal{S}}.
\end{aligned} \tag{6.6}$$

Moreover the commensurability relations in eqs. (2.5) give

$$\begin{aligned}
\mathcal{K}_{\mathcal{S}c} &= \frac{1}{p}[r\mathcal{K}'_{\mathcal{S}c} + s'\mathcal{K}'_{\mathcal{S}d}] = \frac{\mathcal{K}'_{\mathcal{S}}}{p}[rn_0 - s'm_0] \\
\mathcal{K}_{\mathcal{S}d} &= \frac{1}{p}[r'\mathcal{K}'_{\mathcal{S}c} + s\mathcal{K}'_{\mathcal{S}d}] = \frac{\mathcal{K}'_{\mathcal{S}}}{p}[r'n_0 - sm_0].
\end{aligned} \tag{6.7}$$

Finally we get by using eq. (5.13)

$$\sigma_{xy} = \frac{1}{p}\Big[n_f - \sum_{\mathcal{S}}{}' \frac{q}{p}\mathcal{K}'_{\mathcal{S}}\Big] \tag{6.8}$$

where n_f is the number of filled sub-bands.

VII. THE TOPOLOGICAL INVARIANT

The expression of $\sigma_{\rm H}$ as a topological invariant derived by Thouless *et al.* can be obtained from eq. (4.10) by using the sole hypothesis that the chemical potential lies in a gap of the sub-bands. The conditioned sum implies that in such a case any sub-band is either filled or empty. In this situation, by using the periodicity relations in eq. (5.9), one gets

$$\int_{\mathcal{C}} d^2w \partial_k \langle \Upsilon_J^{(w)} | \partial_t \Upsilon_J^{(w)} \rangle = 0. \tag{7.1}$$

Then the Hall resistance (4.10) can be written

$$\begin{aligned}\sigma_{xy} = \Big[& \frac{n_f}{p} - \\ & \frac{i}{2\pi} {\sum_J}' \int_{\mathcal{C}} d^2w \frac{1}{\tau} \int_0^\tau dt \frac{\mathcal{E}_k}{\mathcal{E}^2} \big(\partial_t \langle \Upsilon_J^{(w)} | \partial_k \Upsilon_J^{(w)} \rangle - \partial_k \langle \Upsilon_J^{(w)} | \partial_t \Upsilon_J^{(w)} \rangle \big) \Big].\end{aligned} \tag{7.2}$$

Since Υ depends from w and t only through $w + \tilde{\mathcal{E}} t$, then $\partial_t = \tilde{\mathcal{E}}_j \partial_j$. Finally one gets the Hall conductivity as a topological invariant [1]

$$\begin{aligned}\sigma_{xy} = \Big[& \frac{n_f}{p} - \\ & \frac{i}{2\pi} {\sum_J}' \int_{\mathcal{C}} d^2w \big(\langle \partial_2 \Upsilon_J^{(w)} | \partial_1 \Upsilon_J^{(w)} \rangle - \langle \partial_1 \Upsilon_J^{(w)} | \partial_2 \Upsilon_J^{(w)} \rangle \big) \Big].\end{aligned} \tag{7.3}$$

The expression in Ref. [1] does not contain the first term of the RHS since it is written in the reference frame where the periodic potential is at rest. We prefer the use of eq. (7.3) since the corresponding Hamiltonian is explicitly periodic in w. The proof that the RHS of eq. (7.3) is an integer is similar to the argument given in Ref. [1], with the clause of considering all the sub-bands belonging to a cluster. We use the relations in eqs. (5.9). By imposing monodromy after a tour along the border of the reduced Brillouin zone, one gets

$$-\psi_J(0) - \phi_{J_d}(\frac{d}{p}) + \psi_{J_c}(\frac{c}{p}) + \phi_J(0) = \frac{2\pi}{p} - 2\pi k_{J\sigma} \tag{7.4}$$

where $k_{J\sigma}$ is an integer. Then the conductivity σ_{xy} is (via Stoke's theorem)

$$\begin{aligned}\sigma_{xy} &= \frac{n_f}{p} - \frac{i}{4\pi} {\sum_J}' \int_{\partial\mathcal{C}} dw_l \big(\langle \partial_l \Upsilon_J^{(w)} | \Upsilon_J^{(w)} \rangle - \langle \Upsilon_J^{(w)} | \partial_l \Upsilon_J^{(w)} \rangle \big) \\ &= \frac{n_f}{p} - \frac{i}{4\pi} {\sum_{\mathcal{S}}}' \sum_{J\in\mathcal{S}} \Big\{ \int_{w\in[0,\frac{c}{p}]} dw_l \big[i\partial_l \psi_J(w) + \big(\langle \partial_l \Upsilon_{J_d}^{(w+\frac{d}{p})} | \Upsilon_{J_d}^{(w+\frac{d}{p})} \rangle \\ &- \langle \Upsilon_{J_d}^{(w+\frac{d}{p})} | \partial_l \Upsilon_{J_d}^{(w+\frac{d}{p})} \rangle \big) - \big(\langle \partial_l \Upsilon_J^{(w+\frac{d}{p})} | \Upsilon_J^{(w+\frac{d}{p})} \rangle - \langle \Upsilon_J^{(w+\frac{d}{p})} | \partial_l \Upsilon_J^{(w+\frac{d}{p})} \rangle \big) \big] \\ &+ \int_{w\in[0,\frac{d}{p}]} dw_l \big[-i\partial_l \phi_J(w) - \big(\langle \partial_l \Upsilon_{J_c}^{(w+\frac{c}{p})} | \Upsilon_{J_c}^{(w+\frac{c}{p})} \rangle - \langle \Upsilon_{J_c}^{(w+\frac{c}{p})} | \partial_l \Upsilon_{J_c}^{(w+\frac{c}{p})} \rangle \big) \\ &+ \big(\langle \partial_l \Upsilon_J^{(w+\frac{c}{p})} | \Upsilon_J^{(w+\frac{c}{p})} \rangle - \langle \Upsilon_J^{(w+\frac{c}{p})} | \partial_l \Upsilon_J^{(w+\frac{c}{p})} \rangle \big) \big] \Big\}.\end{aligned} \tag{7.5}$$

After the sum over the sub-bands of a cluster one gets a sum rule

$$\sigma_{xy} = {\sum_{\mathcal{S}}}' \mathcal{K}_{\mathcal{S}\sigma} \qquad \mathcal{K}_{\mathcal{S}\sigma} \equiv \sum_{J\in\mathcal{S}} k_{J\sigma}. \tag{7.6}$$

The last results, together with eq. (6.8), implies

$$\frac{1}{p}[\mathcal{N}_{\mathcal{S}} - \frac{q\mathcal{K}'_{\mathcal{S}}}{p}] = \mathcal{K}_{\mathcal{S}\sigma} \tag{7.7}$$

for every cluster. Since p and q are relative prime numbers, then

$$\mathcal{M}_{\mathcal{S}} \equiv \frac{\mathcal{K}'_{\mathcal{S}}}{p} \tag{7.8}$$

must be an integer and thus we get the Diofantine equation

$$p\ \mathcal{K}_{\mathcal{S}\sigma} + q\ \mathcal{M}_{\mathcal{S}} = \mathcal{N}_{\mathcal{S}} \tag{7.9}$$

where $\mathcal{N}_{\mathcal{S}}$ is the number of sub-bands in the cluster. The validity of the result is general: only the gap condition is required. Once this condition is satisfied, eq. (7.9) should be valid independently from the strength of the periodic potential.

VIII. ADIABATIC APPROXIMATION

In the adiabatic approximation ($\tau \to \infty$) the Floquet state is the instantaneous eigenvector of the Hamiltonian

$$H^{(\Upsilon)}(w + \tilde{\mathcal{E}}t)|\Upsilon_J^{(w)}(t)\rangle = \lambda_J(w + \tilde{\mathcal{E}}t)|\Upsilon_J^{(w)}(t)\rangle \tag{8.1}$$

were the phase is fixed by the Schrödinger equation projected on the vector $\Upsilon_J^{(w)}(t)$

$$\begin{aligned}\langle\Upsilon_J^{(w)}(t)|\partial_t\Upsilon_J^{(w)}(t)\rangle &= -i\langle\Upsilon_J^{(w)}(t)|H^{\Upsilon}(w + \tilde{\mathcal{E}}t)|\Upsilon_J^{(w)}(t)\rangle \\ &= -i\lambda_J(w + \tilde{\mathcal{E}}t).\end{aligned} \tag{8.2}$$

The dependence of Υ from w and t is through the combination $w + \tilde{\mathcal{E}}t$.

The above equation is very important for numerical computation. The phase of the eigenvectors yielded by a computer is usually fixed by requiring that the largest component is real. This choice does not satisfy in general the necessary continuity requirement. Eq. (8.2) allows to fix the phase in the correct way. First one makes a choice of a regular phase on two side vectors of the reduced Brillouin zone. Then the phase of the wave function $\Upsilon_J^{(w)}(t)$ is fixed over the whole reduced Brillouin zone by imposing condition (8.2) over the straight lines parameterized by $w + \tilde{\mathcal{E}}t,\ t \in \mathcal{R}$.

The Floquet energies are given by

$$E_J^{(w)} = \int_0^{\tau} dt\lambda_J(w + \tilde{\mathcal{E}}t) + \alpha_J^{(w)} \tag{8.3}$$

where $\alpha_J^{(w)}$ is the Berry phase. For isolated eigenvalues the adiabatic energy is expected to be periodic over the reduced Brillouin zone

$$\lambda_J(w + m\frac{c}{p} + n\frac{d}{p}) = \lambda_J(w) \tag{8.4}$$

then eq. (5.8) requires

$$\begin{aligned}\alpha_J^{(w+\frac{c}{p})} &= \alpha_J^{(w)} + 2\pi k_{Jc} \\ \alpha_J^{(w+\frac{d}{p})} &= \alpha_J^{(w)} + 2\pi k_{Jd}.\end{aligned} \tag{8.5}$$

By a similar argument one gets also

$$\frac{\partial}{\partial t}\alpha_J^{(w+\tilde{\mathcal{E}}t)} = 0. \tag{8.6}$$

IX. EXAMPLE

We consider the potential

$$V(\mathbf{r}) = v_1 \cos(\frac{q}{p}\tilde{\mathbf{c}}' \cdot \mathbf{r}) + v_2 \cos(\frac{q}{p}\tilde{\mathbf{d}}' \cdot \mathbf{r}). \tag{9.1}$$

This potential is particularly simple since the exponential can be written as

$$\exp[i\tilde{\mathbf{v}} \cdot \mathbf{r}] = S(v)T(-v) \tag{9.2}$$

and therefore the cosine function has a simple expression in terms of unitary operators S and T (eqs. (2.6) and (3.1)). The basis is given by

$$\Phi^{\alpha\beta}_{n_L} = S(w_{\alpha\beta})\Phi^{\alpha_0\beta_0}_{n_L} \qquad n_L = 0, \dots, n_M \tag{9.3}$$

where n_L is Landau level number and α, β have p values and are the pseudo-momenta associated to a finer tiling $(\mathbf{f}, \mathbf{g})$ (with flux one) of $(\mathbf{c}, \mathbf{d})$ lattice. They are fixed by the boundary conditions (2.9). Moreover

$$w_{\alpha\beta} = \frac{1}{2\pi}[(\beta - \beta_0)f - (\alpha - \alpha_0)g]. \tag{9.4}$$

The size of the basis is fixed by n_M, that has to be chosen large enough ($(n_M + \frac{1}{2}) >> |\mathcal{V}|$).

With the given basis one evaluates $U(\tau, 0)$. One extracts the Floquet energies and states by diagonalization of $U(\tau, 0)$.

The direction of the field is chosen to be

$$\tau\tilde{\mathcal{E}} = 2\mathbf{c}' + \mathbf{d}' \tag{9.5}$$

as a compromise between generality and simplicity. The choice of the direction of the electric field determines the periodicity patterns for the mean energy and for the eigenvalues of the evolution operator $U(\tau, 0)$ (or equivalently for the Floquet energies mod(2π)).

The figures 1-8 show the changes in the mean energy (6.1) and in the Floquet energy by varying the period τ and consequently the electric field, according to eq. (9.5). In the whole set of examples we consider the case $q = 2, p = 3$ with

$$\begin{cases} \mathbf{c} = \mathbf{c}' - \mathbf{d}' \\ \mathbf{d} = \mathbf{c}' + \mathbf{d}' \end{cases} \tag{9.6}$$

and use $v_1 = v_2 = 0.5$. We take a basis with Landau number $n_L = 0, \dots, 4$. We show here only the three lowest sub-bands. The relevant factor in eq. (5.12) is then

$$\tau\mathbf{c} \cdot \mathcal{E} = 2\pi\frac{9}{2}. \tag{9.7}$$

The abscissa is a coordinate of the reduced Brillouin zone along the electric field (in the orthogonal direction everything is constant, see eq. (5.11)). The $x = 1.0$ point corresponds to $w = \frac{c}{p}$.

Fig. 1 and 2 give the mean energy and the Berry phase in the adiabatic approximation (i.e. we drop in eq. (8.3) the irrelevant part coming from the instantaneous eigenvalue). The sub-band denoted by circles has $(2\pi)^{-1}\Delta E = -3$ and therefore $\sigma_{\mathrm{H}} = 1$. Stars give $(2\pi)^{-1}\Delta E = 6$ and $\sigma_{\mathrm{H}} = -1$. Boxes give $(2\pi)^{-1}\Delta E = -3$ and $\sigma_{\mathrm{H}} = 1$.

As the period decreases ($\tau = 400$ for Figs. 3 and 4) the gap between the two lowest sub-bands disappears and the periodicity (of mean energy and Floquet energy mod(2π)) is lost. The cluster given circles and stars give together $(2\pi)^{-1}\Delta E = 3$ and $\sigma_{\mathrm{H}} = 0$. Boxes give $(2\pi)^{-1}\Delta E = -3$ and $\sigma_{\mathrm{H}} = 1$.

Figs. 5 and 6 catch the moment ($\tau = 300$) where also the third sub-band start crossing the other two. Fig. 9 shows what happens to the Floquet energies: as the period increases the lines approach each other wedge-wise and eventually merge to form two (almost straight) crossing lines.

Figs. 7 and 8 show the situation at the shorter period $\tau = 100$. By decreasing further the period higher lying sub-bands cross the considered set of three.

For both $\tau = 300$ and $\tau = 100$ the cluster is formed by the three sub-bands: $(2\pi)^{-1}\Delta E = 0$ and $\sigma_{\rm H} = 1$.

We conclude the section with few comments.

- The relation between period and strength of electric field depends on the lattice site as in eq. (3.13). Then one should ask which parameter is relevant for the structure of the sub-bands. We have done few numerical experiments, by keeping the period fixed and by varying the lattice site (thus changing the strength of the electric field). It turns out that the relevant parameter is the strength of the electric field. The change of the lattice site amounts to a change in the periodicity pattern, but the cluster structure remains the same. Fig. 10 provides an instance of this search for the case $\tau = 300$, $\tau\tilde{\mathcal{E}} = 5c' + d'$.

- Consider the situation described by the Figs. 5 and 6. What happens when the chemical potential is $\simeq 0.6$ i.e. in the almost open gap? The condition in eq. (4.1) excludes the parts of the sub-bands above the chemical potential. Thus the filling factor is $\nu_f \simeq 2/3$. An inspection of Fig. 6 shows that $(2\pi)^{-1}\Delta E \simeq 3$ and therefore $\sigma_{\rm H} \simeq 0$ as in the case described by Figs. 3 and 4. However, since there is no gap, the longitudinal conductivity should be appreciably different from zero.

- We have considered a larger basis ($n_M = 12$) in the adiabatic approximation. It is intriguing that for each set of sub-bands coming from the same Landau level the contribution to the transverse Hall conductivity has the pattern (1,-1,1), but it has an exceptional behavior for $n_L = 3$ where it is (-1,3,-1). We have no clues for this anomaly.

X. ACKNOWLEDGMENT

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[1] D.J. Thouless, M. Kohmoto, M.P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).

[2] D. H. Hofstadter, Phys. Rev. B**14**, 2239 (1976).

[3] J. Avron, R. Seiler, and B. Simon, Phys. Rev. Lett. **51**, 51 (1983).

[4] B. Simon, Phys. Rev. Lett. **51**, 2167 (1983).

[5] M. Kohmoto, Ann. Phys. (N.Y.) **160**, 355 (1985).

[6] Q. Niu and D.J. Thouless, J. Phys. A**17**, 2453 (1984).

[7] Q. Niu, D.J. Thouless, and Yong-Shi Wu, Phys. Rev. B**31**, 3372 (1985).

[8] J. Avron and R. Seiler, Phys. Rev. Lett. **54**, 259 (1985).

[9] J. Bellissard, *Ordinary Quantum Hall Effect and non-commutative cohomology*, CPT-86/P.1949 (1986). Proceedings of the Bad Schandau Conference on Localization, Teubner Leipzig, (1987).

[10] N. Imai, K. Ishikawa, T. Matsuyama, and I. Tanaka, Phys. Rev. B**42**, 10610 (1990).

[11] The phenomenology of Quantum Hall Effect in presence of Hofstadter's spectrum is discussed in R.R. Gerhardts, D. Weiss, and U. Wulf, Phys. Rev. B**43**, 5192 (1991).

[12] M.V. Berry, Proc. Roy. Soc. (London) A**392**, 45 (1984).

[13] Throughout the paper we use the units of length, time and energy given by $\lambda = \sqrt{\frac{\hbar c}{eB}}$, $\omega^{-1} = \frac{mc}{eB}$, $\hbar\omega$. We denote two-dimensional vectors either by bold-faced letters $\mathbf{v}$ or by complex numbers $v = v_1 + iv_2$. The dual of a vector is $\tilde{\mathbf{v}} \equiv (-v_2, v_1)$ or $\tilde{v} \equiv iv$.

[14] The representations of the Magnetic Translation Group are discussed in R. Ferrari, Phys. Rev. B**42**, 4598 (1990); R. Ferrari, Int. J. Mod. Phys. B**8**, 529 (1994); R. Ferrari, Int. J. Mod. Phys. B**9**, 3333 (1995).

[15] The condition $[S(v), S(L_j)] = 0$ for $j = 1, 2$ restricts the allowed values for the Magnetic Translations to $v = \frac{1}{g_L}(n_1 L_1 + n_2 L_2)$.

[16] For a short introduction to Floquet energies and for references see J. Bellissard *Quantum systems periodically perturbed in time*, CPT-85/P.1830 (1985). Lectures given at the workshop *Fundamental aspects of quantum theory* Como, Italy, Sept. 2-7, 1985. Eds. V. Gorini and A. Frigerio (Plenum Press, N.Y., 1986).

[17] I. Dana, Y. Avron, and J. Zak, J. Phys. C**18**, L679 (1985).

[18] M. Kohmoto, Phys. Rev. B**16**, 11943 (1989).

[19] M. Kohmoto, B.I. Halperin, and Yong-Shi Wu, Phys. Rev. B**45**, 13488 (1992).

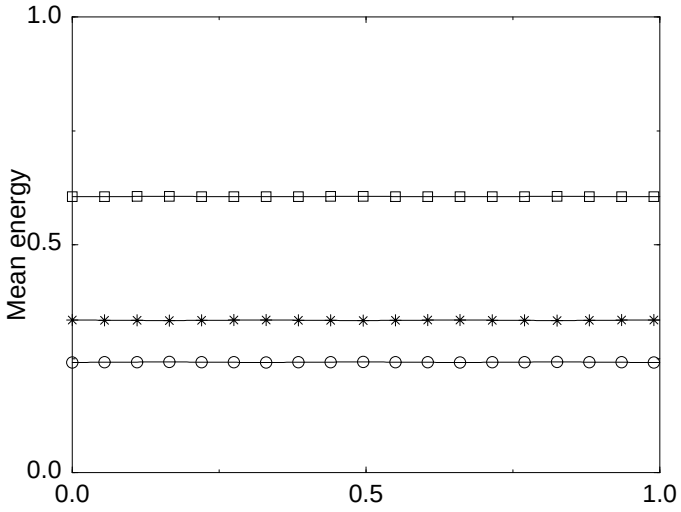

FIG. 1. Mean energy in the adiabatic approximation

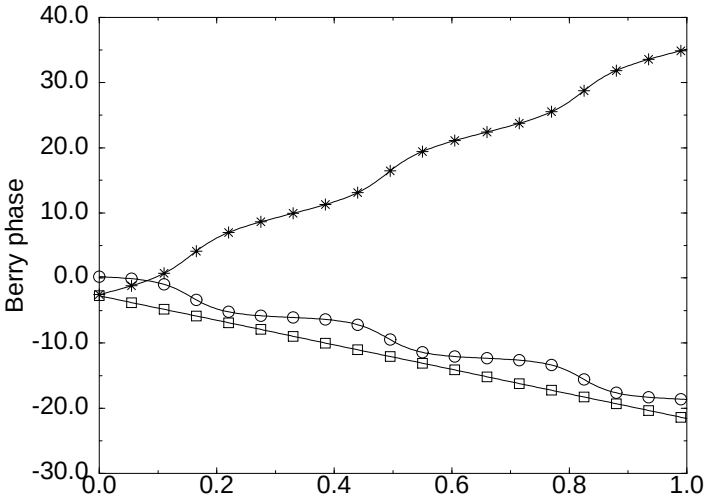

FIG. 2. Berry phase in the adiabatic approximation.

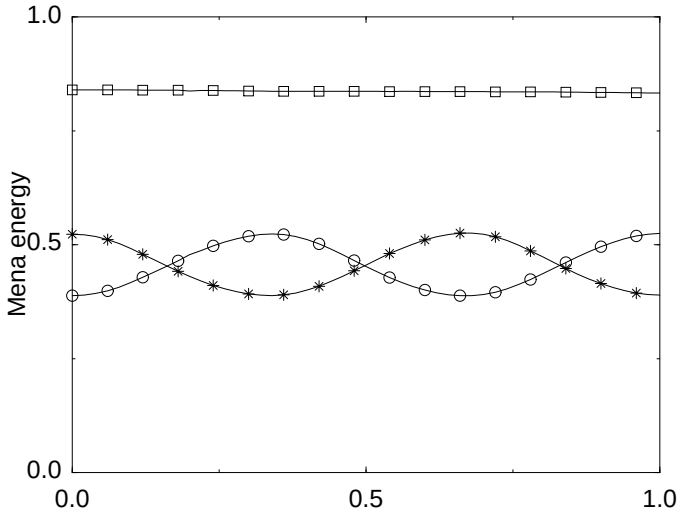

FIG. 3. Mean energy for period $\tau = 400$.

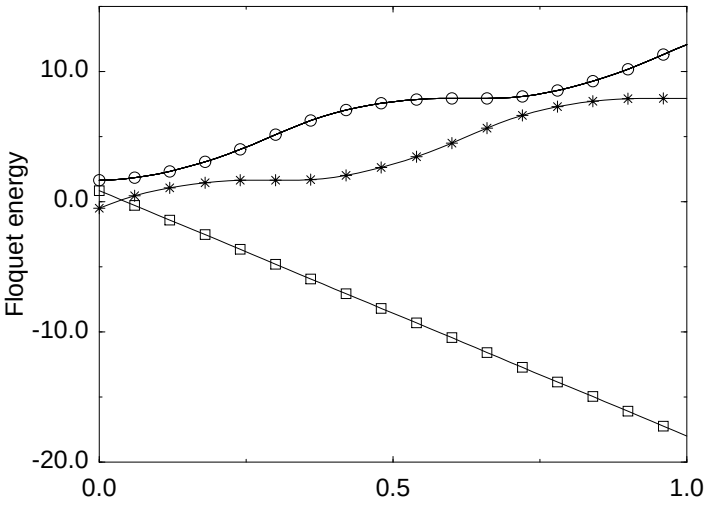

FIG. 4. Floquet energy for period $\tau = 400$.

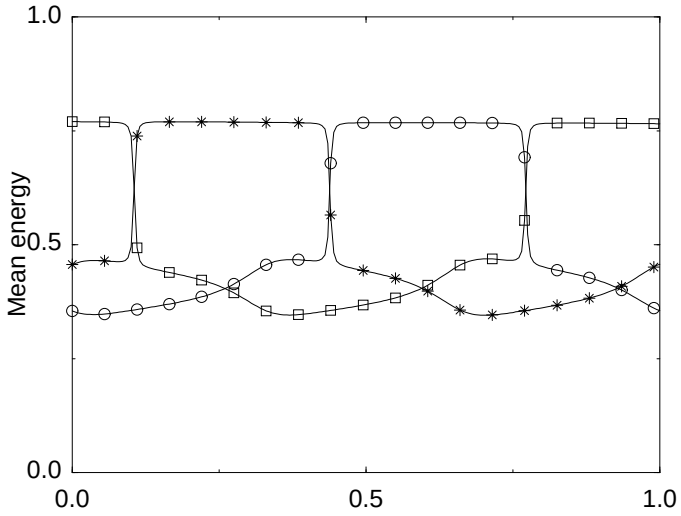

FIG. 5. Mean energy for period $\tau = 300$.

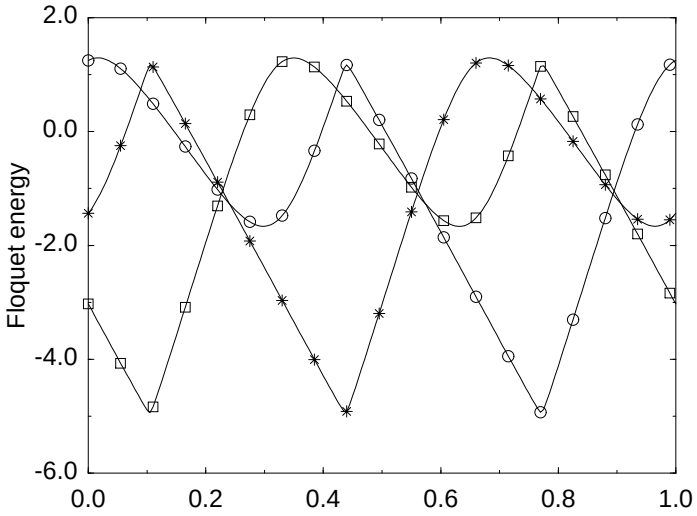

FIG. 6. Floquet energy for period $\tau = 300$.

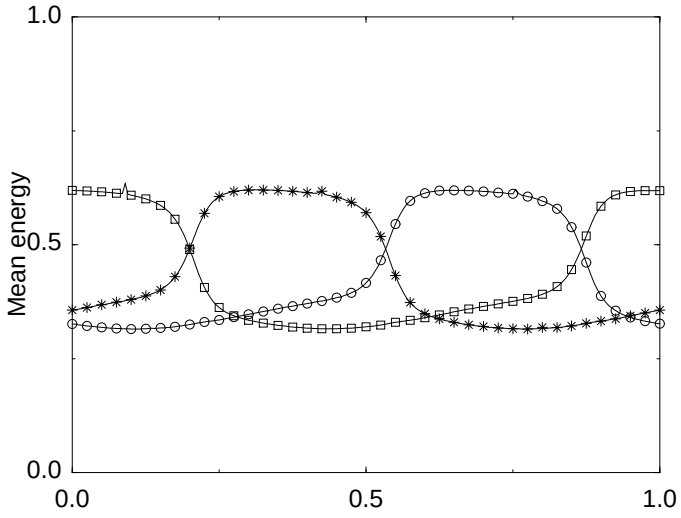

FIG. 7. Mean energy for period $\tau = 100$.

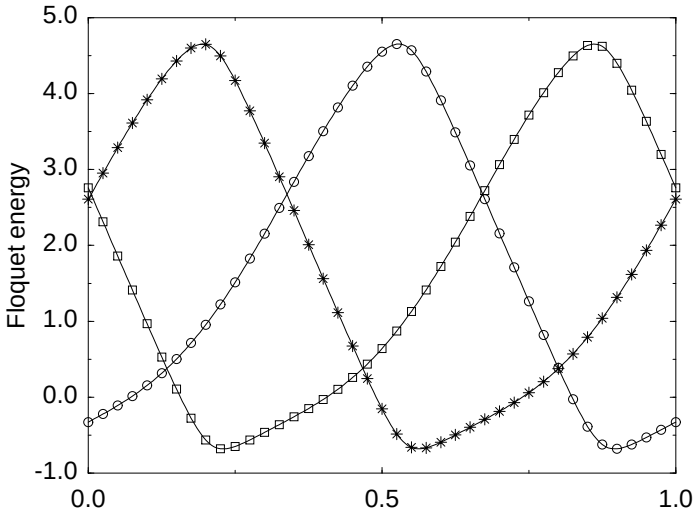

FIG. 8. Floquet energy for period $\tau = 100$.

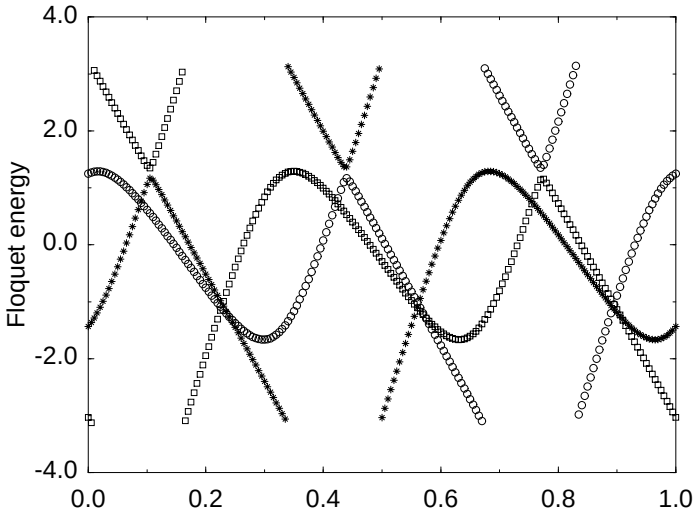

FIG. 9. Floquet energy (mod2π) for period $\tau = 300$.

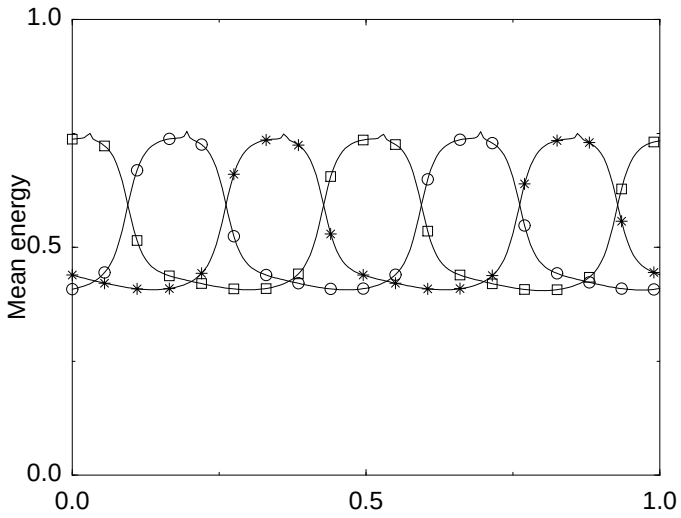

FIG. 10. Mean energy for period $\tau = 300$, $\tau\tilde{\mathcal{E}} = 5\ c' + d'$.